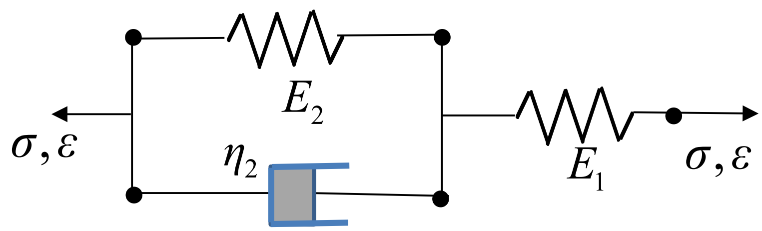
Exercise 1: Derive the rheological equation for the configuration below.



Solution:

Due to a series setting the spring and Kelvin-Voight model are subjected to the same stress $\sigma_1 = \sigma_2 = \sigma$ and the strains expresses as,

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \tag{a}$$

For the Kelvin's model we have,

$$\sigma = E_2 \varepsilon_2 + \eta_2 \dot{\varepsilon}_2 \tag{b}$$

and the spring,

$$\varepsilon_1 = \frac{\sigma}{E_1} \tag{c}$$

To establish the constitutive equation, we proceed as follow.

From (a) we have,

$$\varepsilon_2 = \varepsilon - \varepsilon_1 \tag{d}$$

Combining (c) and (d) we obtain,

$$\varepsilon_2 = \varepsilon - \varepsilon_1 = \varepsilon - \sigma / E_1 \quad \Rightarrow \quad \dot{\varepsilon}_2 = \dot{\varepsilon} - \dot{\sigma} / E_1 \tag{e}$$

Introducing (e) in (b) we have,

$$\begin{aligned} &\sigma = E_2(\varepsilon - \sigma / E_1) + \eta_2 \left(\dot{\varepsilon} - \dot{\sigma} / E_1\right) \quad \Rightarrow \quad \sigma = E_2 \varepsilon - \sigma \frac{E_2}{E_1} + \eta_2 \dot{\varepsilon} - \dot{\sigma} \frac{\eta_2}{E_1} \\ &\sigma + \sigma \frac{E_2}{E_1} + \dot{\sigma} \frac{\eta_2}{E_1} = E_2 \varepsilon + \eta_2 \dot{\varepsilon} \quad \Rightarrow \quad \sigma \left(1 + \frac{E_2}{E_1}\right) + \dot{\sigma} \frac{\eta_2}{E_1} = E_2 \varepsilon + \eta_2 \dot{\varepsilon} \end{aligned} \tag{f}$$

Rearranging the last relation, we obtain the final equation, conventionally called the equation of state of a *standard solid*, Kelvin-Voight representation.

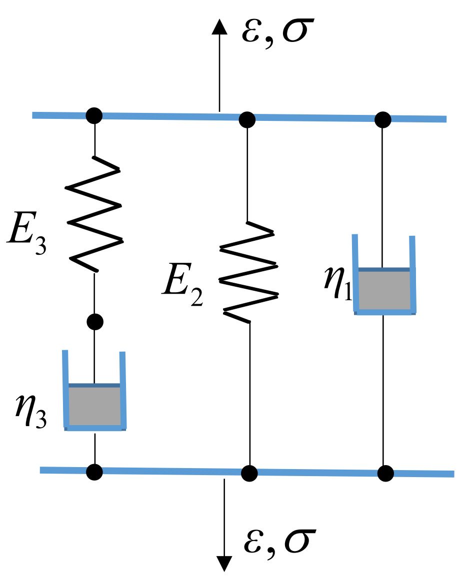
$$\sigma + \dot{\sigma}\frac{\eta_2}{E_1 + E_2} = \frac{E_1 E_2}{E_1 + E_2}\varepsilon + \frac{E_1 \eta_2}{E_1 + E_2}\dot{\varepsilon} \qquad \text{(g)}$$

Note that (g) is a particular case of equation (B.13).

Exercise 2: Verify that,

$$\eta_1 \eta_3 \ddot{\varepsilon} + (\eta_3 E_3 + E_2 \eta_3 + \eta_1 E_3)\dot{\varepsilon} + E_2 E_3 \varepsilon = \eta_3 \dot{\sigma} + E_3 \sigma$$

is the constitutive, or rheological, equation for the configuration shown in the figure below[1],



Solution:

In this configuration, all elements are subjected to the same strain,

$$\varepsilon = \varepsilon_1 = \varepsilon_2 = \varepsilon_3 \qquad \text{(a)}$$

[1] Note that the dots on the strain and stresses indicate derivatives with respect to time

The stresses are,

$$\sigma_1 + \sigma_2 + \sigma_3 = \sigma \tag{b}$$

Thus, we have,

$$\sigma_1 = \eta_1 \dot{\varepsilon}\,, \quad \sigma_2 = E_2 \varepsilon\,, \quad \dot{\varepsilon} = \frac{\dot{\sigma}_3}{E_3} + \frac{\sigma_3}{\eta_3} \tag{c}$$

These equations are combined in the following way,

$$\sigma_3 = \sigma - \sigma_1 - \sigma_2 = \sigma - \eta_1 \dot{\varepsilon} - E_2 \varepsilon \quad \Rightarrow \quad \dot{\sigma}_3 = \dot{\sigma} - \eta_1 \ddot{\varepsilon} - E_2 \dot{\varepsilon} \tag{d}$$

Introduce this last two expressions in the third of (c),

$$\dot{\varepsilon} = \frac{\dot{\sigma} - \eta_1 \ddot{\varepsilon} - E_2 \dot{\varepsilon}}{E_3} + \frac{\sigma - \eta_1 \dot{\varepsilon} - E_2 \varepsilon}{\eta_3} \tag{e}$$

This equation can be rewritten as,

$$\dot{\varepsilon} = \frac{\eta_3}{\eta_3 E_3}\left[\dot{\sigma} - \eta_1 \ddot{\varepsilon} - E_2 \dot{\varepsilon}\right] + \frac{E_3}{\eta_3 E_3}\left[\sigma - \eta_1 \dot{\varepsilon} - E_2 \varepsilon\right]$$

This expression is rewritten to arrive at the compact form,

$$\eta_3 E_3 \dot{\varepsilon} = \eta_3 \dot{\sigma} - \eta_1 \eta_3 \ddot{\varepsilon} - E_2 \eta_3 \dot{\varepsilon} + E_3 \sigma - \eta_1 E_3 \dot{\varepsilon} - E_2 E_3 \varepsilon$$

$$\eta_1 \eta_3 \ddot{\varepsilon} + (\eta_3 E_3 + E_2 \eta_3 + \eta_1 E_3)\dot{\varepsilon} + E_2 E_3 \varepsilon = \eta_3 \dot{\sigma} + E_3 \sigma \tag{f}$$